ISRAEL JOURNAL OF MATHEMATICS, Vol. 59, No. 1, 1987

GENERIC PROPERTIES OF INVARIANT MEASURES FOR SIMPLE PIECEWISE MONOTONIC TRANSFORMATIONS

BY

FRANZ HOFBAUER

Institut für Mathematik, Universität Wien, Strudlhofgasse 4, A-1090 Wien, Austria

ABSTRACT

We endow the set of all invariant measures of topologically transitive subsets L of certain piecewise monotonic transformations on [0, 1] with the weak topology. We show that the set of periodic orbit measures is dense, that the sets of ergodic, of nonatomic, and of measures with support L are dense G_δ-sets, that the set of strongly mixing measures is of first category, and that the set of measures with zero entropy contains a dense G_δ-set.

Introduction

A map $T: [0, 1] \to [0, 1]$ is called piecewise monotonic, if there is a finite partition $\mathscr{Z}$ of [0, 1] into subintervals, such that $T \mid Z$ is continuous and monotone for all $Z \in \mathscr{Z}$. Simple examples are unimodal and monotonic mod 1 transformations. T is called unimodal, if it is continuous, if there is a $c \in (0, 1)$ such that $T \mid [0, c]$ is increasing and $T \mid [c, 1]$ is decreasing, and if $T(c) = 1$ and $T(1) = 0$. T is called monotonic mod 1, if there is an increasing continuous function $f: [0, 1) \to \mathbf{R}$ with $T(x) = f(x)$ mod 1 for $x \in [0, 1)$ and with $T(1) = T(1-)$. We consider in this paper only unimodal maps and monotonic mod 1 transformations with the additional property that $f(0) \in [0, 1)$ and $f(1-) \in (1, 2]$, although one can prove the same results with similar, but more complicated methods also for other simple piecewise monotonic maps, in particular for all monotonic mod 1 transformations. A dynamical system (X, T) is a continuous map T on a compact metric space X. In general, a piecewise monotonic map has finitely many discontinuities. By a slight modification (cf. §4 below), it becomes a dynamical system.

Received October 15, 1985

As we investigate invariant measures of a dynamical system, we can restrict ourselves to the center of the dynamical system, which contains the support of all invariant measures. It is known (cf. [6] and [9]) that the center of unimodal and monotonic mod 1 transformations can be written as $\bigcup_{1 \leqq i \leqq n} L_i$, where $n \leqq \infty$ and the L_i are topologically transitive with $L_i \cap L_j = \emptyset$, if $i \neq j$, except $i = n < \infty$ and $j = n - 1$. Then this intersection may be finite. It may happen that L_n has only nonperiodic orbits. It happens, if $n = \infty$. Then the results of this paper clearly cannot hold for L_n. If L_i is an isolated periodic orbit, then the results of this paper are trivial. The other L_i are exactly those which satisfy $h_{\text{top}}(T \mid L_i) > 0$. For these L_i we shall prove Theorem 2 below. If $i < n$, then L_i is isomorphic to a finite type subshift, for which Theorem 2 is already known (cf. [1]). The new result is that about L_n.

The method, which was introduced for the investigation of generic properties of invariant measures for axiom A diffeomorphisms, is the specification property (cf. §21 of [1]). For our purposes we propose a weaker form of the specification property in §1 below. Then, in §3, we are able to prove

THEOREM 1. *Suppose the dynamical system (X, T) has the specification property defined in §1. Let U be a nonempty open subset of the set $M(X, T)$ of all T-invariant probability measures on X with respect to weak topology. Then the periodic points $x \in X$ with $m_x \in U$ are dense in X, where m_x is the T-invariant measure concentrated on the periodic orbit of x.*

In §2 and §4, we show then that this specification property holds for shift spaces, which are isomorphic to the dynamical systems $(L, T \mid L)$, where L is a topologically transitive subset of one of the piecewise monotonic maps we consider in this paper, such that $h_{\text{top}}(T \mid L) > 0$. This gives that the assertion of Theorem 1 holds for such $(L, T \mid L)$. The same proofs as in [1] and [10] show then

THEOREM 2. *Let $(L, T \mid L)$ be as above. With respect to weak topology we have then:*

(i) *The set of measures concentrated on periodic orbits is dense in $M(L, T \mid L)$.*

(ii) *The set of ergodic measures is a dense G_δ-subset of $M(L, T \mid L)$.*

(iii) *The set of nonatomic invariant measures is a dense G_δ-subset of $M(L, T \mid L)$.*

(iv) *The set of measures with support L is a dense G_δ-subset of $M(L, T \mid L)$.*

(v) *The set of strongly mixing measures is of first category in $M(L, T \mid L)$.*

(vi) *The set of measures with zero entropy contains a dense G_δ-subset of* $M(L, T \mid L)$.

In fact, all assertions of this theorem except (vi) hold for all dynamical systems (X, T) which satisfy the assertion of Theorem 1 and $h_{\text{top}}(X, T) > 0$. The proofs are as in [1]. One needs that for every nonempty open subset U of $M(X, T)$ and every $n \in \mathbf{N}$ there is a periodic point $x \in X$ with $T^i(x) \neq x$ for $1 \leqq i < n$ and $m_x \in U$. One gets this from Theorem 1 as follows: As $h_{\text{top}}(X, T) > 0$, there is a $y \in X$ such that the points $T^i(y)$ for $0 \leqq i < n$ are different. One finds open neighbourhoods Q_i of $T^i(y)$ for $0 \leqq i < n$, which are pairwise disjoint. Then $\bigcap_{i=0}^{n-1} T^{-i}(Q_i)$ is open and nonempty. Hence it contains a periodic point x with $m_x \in U$ by Theorem 1. As $T^i(x) \in Q_i$ for $0 \leqq i < n$, we have $T^i(x) \neq x$ for $1 \leqq i < n$. For (iv), even weaker conditions are sufficient (cf. (21.11) and (21.12) of [1]). By results of [2] and [3], such conditions hold for every topologically transitive $(L, T \mid L)$ with $h_{\text{top}}(T \mid L) > 0$ of every piecewise monotonic transformation and not only for simple ones. The proof of (vi) is as in [10]. Instead of the results about partitions of axiom A diffeomorphisms used there, one can use results of §3 of [3].

In §21 of [1] also the sets $V_T(x) = \{m \in M(X, T)$: m is a limit point of $(1/n) \sum_{i=0}^{n-1} \delta_{T^i(x)}\}$ are investigated. For example, it is shown that for every nonempty closed connected subset V of $M(X, T)$ there is a dense subset of $x \in X$ with $V_T(x) = V$, if (X, T) has the specification property. The proofs of these results can be modified such that they hold also if one uses the weaker specification propery of §1 of this paper.

§1. The specification property

It is not quite easy to give a nice definition of a suitable specification property. We choose the following definition, which is adapted to piecewise monotonic transformations. Let (X, T) be a dynamical system, i.e. T is a continuous map on the compact metric space X. We denote the distance on the metric space X by dist. We call a pair $(x, l) \in X \times \mathbf{N}$ an orbit segment, thinking that it represents $\{T^i(x): 0 \leqq i < l\}$. We suppose that there is a map $d: X \times \mathbf{N} \times X \times \mathbf{N} \times (0, 1) \to \mathbf{N}$ such that the following holds: For finitely many given orbit segments $(y_0, l_0), (y_1, l_1), \ldots, (y_t, l_t)$ and a given $\delta \in (0, 1)$ there exist a periodic point $z \in X$ and integers $u_0 = 0, u_1, \ldots, u_t, u_{t+1} = p$ with $T^p(z) = z$ and $u_i + l_i \leqq u_{i+1}$ for $0 \leqq i \leqq t$ such that (1.1) and (1.2) below hold, where we set $(y_{t+1}, l_{t+1}) = (y_0, l_0)$:

(1.1) $$u_{i+1} - (u_i + l_i) \leqq d(y_i, l_i, y_{i+1}, l_{i+1}, \delta) \qquad \text{for } 0 \leqq i \leqq t,$$

(1.2) $\quad \operatorname{dist}(T^{u_i+j}(z), T^j(y_i)) < \delta \qquad$ for $0 \leqq j < l_i$ and $0 \leqq i \leqq t$.

This definition says that parts of the periodic orbit of z approximate the given orbit segments well (cf. (1.2)) and that the parts in between are bounded by d (cf. (1.1)). For example, such a map d can be shown to exist, if (X, T) is locally eventually onto, i.e. if for every open nonempty subset U of X there is an $n \in \mathbf{N}$ with $T^n(U) = X$. Examples of shift spaces, for which d exists, are considered in §2. Now we say that (X, T) has the specification property, if d can be defined in such a way, that for all $x \in X$, for all $\varepsilon > 0$, for all $\delta > 0$, and for all $H \in \mathbf{N}$, there are an integer r and integers $0 = w_0 \leqq v_1 < w_1 \leqq v_2 < w_2 \leqq \cdots \leqq v_r < w_r$ such that (1.3), (1.4) and (1.5) below hold:

(1.3) $$w_r \geqq H,$$

(1.4) $$\sum_{i=1}^{r} (v_i - w_{i-1}) < \varepsilon w_r,$$

(1.5) $$d(T^{v_i}(x), w_i - v_i, T^{v_i}(x), w_i - v_i, \delta) < \varepsilon(w_i - v_i) \qquad \text{for } 1 \leqq i \leqq r.$$

This definition says that we can partition arbitrary long initial pieces (cf. (1.3)) of the orbit of any $x \in X$ into orbit segments $(T^{v_i}(x), w_i - v_i)$ with small gaps in between (cf. (1.4)), such that these orbit segments behave well with respect to the function d (cf. (1.5) and the proof of Theorem 1 in §3).

§2. Shift spaces defined by a graph

In this section we show how one can prove the specification property for certain shift spaces over a finite alphabet S. Let $\mathscr{C}$ be a finite or countable irreducible oriented graph (we denote both the graph and the set of its vertices by $\mathscr{C}$) and let Ψ: $\mathscr{C} \to S$ be a map. The shift space X is then defined as the set of all sequences $\Psi(C_1)\Psi(C_2)\cdots \in S^{\mathbf{N}}$, where $C_1C_2\cdots$ is a path in $\mathscr{C}$. A sequence of vertices $C_i \in \mathscr{C}$ is called a path if C_{i+1} is a successor of C_i for $i \geqq 1$. Clearly X is invariant under the shift transformation σ. For the graphs we consider in this paper, X is also closed.

Now we define the map d: $X \times \mathbf{N} \times \mathbf{X} \times \mathbf{N} \times (0, 1) \to \mathbf{N}$ which will enable us to show the specification property under certain assumptions. Since $\mathscr{C}$ is irreducible, for $C, D \in \mathscr{C}$ there is a path leading from C to D. Let $q(C, D)$ be the minimum of the lengths of all these paths. For every orbit segment (x, l) there is at least one path $C_1C_2\cdots C_l$ of length l in $\mathscr{C}$, such that $\Psi(C_1)\Psi(C_2)\cdots\Psi(C_l)$ are the first l symbols of x. Now assign to (x, l) one of these paths $C_1C_2\cdots C_l$ for which $q(C_l, C_1)$ is minimal. For a given $\delta > 0$ fix n

such that for $x, x' \in X$ one has $\mathrm{dist}(x, x') < \delta$, if the first n symbols of x and x' coincide. If (x, l) and (y, m) are given orbit segments, let C be the last vertex of the path of length $l + n$ assigned to $(x, l + n)$ and let D be the first vertex of the path of length $m + n$ assigned to $(y, m + n)$. Then set $d(x, l, y, m, \delta) = n + q(C, D)$. Now we show that (1.1) and (1.2) hold. We define z as $\Psi(D_1)\Psi(D_2)\cdots$ for some closed path $D_1D_2\cdots$. That $D_1D_2\cdots$ is closed, means that there is a $p \in \mathbf{N}$ with $D_{p+i} = D_i$ for all $i \geqq 1$, which implies $T^p(z) = z$. Using the definitions of n and $q(C, D)$, it is easy to check that (1.1) and (1.2) hold, if one chooses $D_1D_2\cdots$ as follows: For given orbit segments $(y_0, l_0), (y_1, l_1), \ldots, (y_t, l_t)$, let $D_1D_2\cdots$ be the closed path one gets, if one runs through the path assigned to $(y_0, l_0 + n)$, then goes on a shortest path to the vertex at which the path assigned to $(y_1, l_1 + n)$ begins, runs through this path, goes on a shortest path to the vertex, at which the path assigned to $(y_2, l_2 + n)$ begins, and so on, until one runs through the path assigned to $(y_t, l_t + n)$ and goes back on a shortest path to the vertex, at which the path assigned to $(y_0, l_0 + n)$ begins.

Now we can show

LEMMA 1. *Let (X, σ) be a shift space defined by an oriented irreducible graph $\mathscr{C}$ as above. If every path $C_1C_2\cdots$ of infinite length in $\mathscr{C}$ satisfies either* (i) *or* (ii) *below, then (X, σ) has the specification property.*

(i) *There are a $D \in \mathscr{C}$ and infinitely many $i \in \mathbf{N}$ such that D is a successor of C_i.*

(ii) *There are an integer c and integers $i_0 < i_1 < i_2 < \cdots$ such that $i_k - i_{k-1} \to \infty$ for $k \to \infty$ and such that for every $k > 0$ there is an $N(k)$ with $0 \leqq N(k) < k$, an $l \in \mathbf{N}$ with $i_k - i_{N(k)} - c \leqq l \leqq i_k - i_{N(k)} + c$ and a closed path $D_1D_2\cdots D_lD_1D_2\cdots D_lD_1D_2\cdots$ in $\mathscr{C}$ with $\Psi(D_j) = \Psi(C_{i_{N(k)}+j})$ for $1 \leqq j \leqq i_k - i_{N(k)} - c$.*

PROOF. In order to show the specification property, we define d as was done above and fix an $x \in X$, an $\varepsilon > 0$, a $\delta > 0$ and an $H \in \mathbf{N}$. For δ we fix an $n \in \mathbf{N}$ as above. By definition of X, there is a path $C_1C_2\cdots$ in $\mathscr{C}$ with $x = \Psi(C_1)\Psi(C_2)\cdots$.

Suppose first that (i) holds for the path $C_1C_2\cdots$. By (i) we find an i such that D is a successor of C_i, such that $i \geqq H + n$ and such that $n + q(D, C_1) + 1 < \varepsilon(i - n)$. Set $r = 1$, $v_1 = 0$ and $w_1 = i - n$. Then $i \geqq H + n$ implies (1.3). Furthermore (1.4) is trivial, since $r = 1$ and $v_1 = w_0 = 0$. Because of $n + q(D, C_1) + 1 < \varepsilon(i - n) = \varepsilon w_1$ we get (1.5) from $q(C_i, C_1) \leqq q(D, C_1) + 1$, which holds, as D is a successor of C_i, and from $d(x, w_1, x, w_1, \delta) \leqq n +$

$q(C_i, C_1)$, which holds, since $\Psi(C_1)\Psi(C_2)\cdots\Psi(C_i)$ are the first $i = w_1 + n$ symbols of x and hence $C_1C_2\cdots C_i$ is a possible candidate to be assigned to $(x, w_1 + n)$.

Now suppose that (ii) holds for the path $C_1C_2\cdots$. Since $i_k - i_{k-1} \to \infty$ for $k \to \infty$, there is a $u \geqq 1$ with

$$n + 2c \leqq \varepsilon(i_m - i_{m-1} - n - c) \qquad \text{for } m \geqq u \tag{2.1}$$

and an s with $i_s - n - c \geqq H$ and with

$$i_u + (n + c)(s - 1) \leqq \varepsilon(i_s - n - c). \tag{2.2}$$

Set $s_0 = s$. If s_j is defined, set $s_{j+1} = N(s_j)$. As $s_{j+1} < s_j$, we find an $r \leqq s$ such that $s_r \leqq u < s_{r-1}$. Now we set $v_k = i_{s_{r-k+1}}$ and $w_k = i_{s_{r-k}} - n - c$ for $1 \leqq k \leqq r$. Since $w_r = i_s - n - c$, we get (1.3) from $i_s - n - c \geqq H$. Furthermore $v_{k+1} - w_k = n + c$ for $1 \leqq k \leqq r - 1$ and $v_1 = i_{s_r} \leqq i_u$, hence (1.4) follows from (2.2), since $r \leqq s$ and $w_r = i_s - n - c$. Finally for $1 \leqq k \leqq r$ there is a closed path $D_1D_2\cdots D_lD_1D_2\cdots$ such that

$$w_k - v_k + n \leqq l \leqq w_k - v_k + n + 2c \tag{2.3}$$

and

$$\Psi(D_j) = \Psi(C_{v_k+j}) \qquad \text{for } 1 \leqq j \leqq w_k - v_k + n \tag{2.4}$$

since $w_k + n + c = i_{s_{r-k}}$ and $v_k = i_{s_{r-k+1}} = i_{N(s_{r-k})}$.

By (2.4) the path $D_1D_2\cdots D_{w_k-v_k+n}$ is a possible candidate to be assigned to $(T^{v_k}(x), w_k - v_k + n)$, which gives

$$d(T^{v_k}(x), w_k - v_k, T^{v_k}(x), w_k - v_k, \delta) \leqq n + l - (w_k - v_k + n) \tag{2.5}$$

since $D_{w_k-v_k+n+1}D_{w_k-v_k+n+2}\cdots D_l$ is a path of length $l - (w_k - v_k + n)$ from $D_{w_k-v_k-n}$ to D_1. By (2.3) we get $n + l - (w_k - v_k + n) \leqq n + 2c$, and by (2.1) we get $n + 2c \leqq \varepsilon(i_m - i_{m-1} - n - c)$ for $m = s_{r-k}$, since $s_{r-k} \geqq u$, if $k \geqq 1$. As $N(m) \leqq m - 1$, we get $n + 2c \leqq \varepsilon(i_m - n - c - i_{N(m)}) = \varepsilon(w_k - v_k)$, and (1.5) follows from (2.5). □

We conclude §2 with two examples. First consider the case where $\mathscr{C}$ is finite. This includes all topologically transitive finite type subshifts and is essentially the case cosidered in [1] and [10]. For every path of infinite length in $\mathscr{C}$ there is a $C \in \mathscr{C}$, which occurs infinitely often in this path, as $\mathscr{C}$ is finite. If D is a successor of C, we get (i) of Lemma 1 with this D. Hence (i) of Lemma 1 holds for all paths.

The second example we consider is the β-shift, whose set of invariant measures is investigated in [11]. Choose $n \in \mathbf{N}$ such that $n < \beta \leqq n+1$, set $S = \{0, 1, \ldots, n\}$ and let $e_1 e_2 e_3 \cdots \in S^{\mathbf{N}}$ be the β-expansion of 1. If $e_i e_{i+1} e_{i+2} \cdots$ is periodic for some i or if there is an i with $e_j = 0$ for all $j \geqq i$, then the β-shift can be considered as a finite type subshift. Otherwise set $\mathscr{C} = \{E_1, \ldots, E_n\} \cup \{A_i\colon i \geqq 1\}$ and insert arrows from A_i to A_{i+1} for $i \geqq 1$, from E_i to E_j for $1 \leqq i, j \leqq n$, from E_i to A_1 for $1 \leqq i \leqq n$ and from A_i to E_j for $1 \leqq j \leqq e_i$ and $i \geqq 1$. If $\Psi(E_i) = i - 1$ for $1 \leqq i \leqq n$ and $\Psi(A_i) = e_i$ for $i \geqq 1$, then $\mathscr{C}$ and Ψ define a shift space as described at the beginning of §2, which is the β-shift (cf. [4]). As $e_i > 0$ for infinitely many i, there are infinitely many A_i, which have E_1 as successor. If $C_1 C_2 \cdots$ is a path in $\mathscr{C}$, then either it contains some E_j infinitely often and we have (i) of Lemma 1 as above, or there are j and k with $C_{j+i} = A_{k+i}$ for $i \geqq 0$ and $C_1 C_2 \cdots$ contains infinitely many elements, which have E_1 as successor. This is again (i) of Lemma 1.

§3. A property implied by specification

In this section we prove

THEOREM 1. *Suppose the dynamical system (X, T) has the specification property. Let U be a nonempty open subset of $M(X, T)$ and let V be a nonempty open subset of X. Then there is a periodic point $z \in V$ with $m_z \in U$.*

The proof of Theorem 1 is adapted from [1]: For the open subset U of $M(X, T)$ we find a $\mu \in M(X, T)$, an $\eta > 0$ and a finite subset $\mathscr{F}$ of $C(X, \mathbf{R})$, such that

$$W(\mu, \mathscr{F}, \eta) := \{\nu \in M(X, T)\colon |\textstyle\int f d\mu - \int f d\nu| < \eta \ \forall f \in \mathscr{F}\} \subset U.$$

Set $F = \max\{\|f\|_\infty\colon f \in \mathscr{F}\}$, set $\varepsilon = \eta/(10F + 3)$ and choose $\delta > 0$ and $w \in X$ such that

$$\text{(3.1)} \qquad \operatorname{dist}(x, y) < \delta \Rightarrow |f(x) - f(y)| < \varepsilon \qquad \forall f \in \mathscr{F}$$

and such that $\{x \in X\colon \operatorname{dist}(x, w) < \delta\} \subset V$.

Set $S_j f(x) = \sum_{i=0}^{j-1} f(T^i x)$. By the ergodic theorem there is a subset Q of X with $\mu(Q) = 1$ such that for every $f \in \mathscr{F}$ there is an $f^* \in L^1_\mu$ with $\int f^* d\mu = \int f d\mu$ and $(1/j) S_j f(x) \to f^*(x)$ for all $x \in Q$. Since $\|f^*\|_\infty \leqq \|f\|_\infty \leqq F$, we find a finite partition $\{Q_1, Q_2, \ldots, Q_n\}$ of Q such that $f^* \mid Q_i$ has oscillation $< \varepsilon$ for $1 \leqq i \leqq n$ and for all $f \in \mathscr{F}$. Choose $x_i \in Q_i$. Then

$$\left| \sum_{i=1}^{n} \mu(Q_i) f^*(x_i) - \int f d\mu \right| < \varepsilon \qquad \text{for all } f \in \mathscr{F}.$$

We can approximate $\mu(Q_i)$ by $q_i \in \mathbf{Q}$ such that for all $f \in \mathcal{F}$

$$\left| \int f d\mu - \sum_{i=1}^{n} q_i f^*(x_i) \right| < \varepsilon \quad \text{and} \quad \sum_{i=1}^{n} q_i = 1. \tag{3.2}$$

Next choose H such that

$$\left| \frac{1}{j} S_j f(x_i) - f^*(x_i) \right| < \varepsilon \qquad \text{for } j \geqq H, \quad \text{for } 1 \leqq i \leqq n, \quad \text{and for all } f \in \mathcal{F}.$$

We apply the specification property to x_i, where $1 \leqq i \leqq n$, using $\varepsilon > 0$, $\delta > 0$ and H as above. For every i we get an integer $r = r_i$ and integers v_j, w_j for $1 \leqq j \leqq r$ with $w_0 = 0 \leqq v_1 < w_1 \leqq v_2 < w_2 \leqq \cdots \leqq v_r < w_r$ such that (1.3), (1.4) and (1.5) hold. Set $x_{ij} = T^{v_j}(x_i)$, $m_{ij} = w_j - v_j$ and $k_i = \Sigma_{j=1}^{r} m_{ij}$. Then we have the following estimations:

$$\left| f^*(x_i) - \frac{1}{k_i} \sum_{j=1}^{r} S_{m_{ij}} f(x_{ij}) \right| \leqq A + B + C$$

where

$$A := \left| f^*(x_i) - \frac{1}{w_r} S_{w_r} f(x_i) \right| < \varepsilon \qquad \text{by (1.3)},$$

$$B := \left| \frac{1}{w_r} S_{w_r} f(x_i) - \frac{1}{w_r} \sum_{j=1}^{r} S_{m_{ij}} f(x_{ij}) \right| = \frac{1}{w_r} \left| \sum_{j=1}^{r} \sum_{l=w_{j-1}}^{v_j - 1} f(T^l x_i) \right|$$

$$\leqq \frac{1}{w_r} F \sum_{j=1}^{r} (v_j - w_{j-1}) \leqq \varepsilon F \qquad \text{by (1.4)},$$

$$C := \left| \frac{1}{w_r} \sum_{j=1}^{r} S_{m_{ij}} f(x_{ij}) - \frac{1}{k_i} \sum_{j=1}^{r} S_{m_{ij}} f(x_{ij}) \right| \leqq \left| \frac{1}{w_r} - \frac{1}{k_i} \right| F \sum_{j=1}^{r} m_{ij}$$

$$= \frac{|w_r - k_i|}{w_r k_i} F k_i = \frac{1}{w_r} \sum_{j=1}^{r} (v_j - w_{j-1}) F \leqq \varepsilon F \qquad \text{by (1.4)}.$$

Using $\Sigma_{i=1}^{n} q_i = 1$ (cf. (3.2)) this gives for all $f \in \mathcal{F}$

$$\left| \sum_{i=1}^{n} q_i f^*(x_i) - \sum_{i=1}^{n} \frac{q_i}{k_i} \sum_{j=1}^{r_i} S_{m_{ij}} f(x_{ij}) \right| \leqq \varepsilon(2F + 1). \tag{3.3}$$

For the next step set $D = \max d(x, l, y, m, \delta)$, where the maximum is taken over all $(x, l), (y, m) \in \{(x_{ij}, m_{ij}), (w, 1) \colon 1 \leqq i \leqq n,\ 1 \leqq j \leqq r_i\}$. Fix $K \geqq D/\varepsilon$. Since $q_i \in \mathbf{Q}$, we find integers c and c_i such that $q_i/k_i = c_i/c$ for $1 \leqq i \leqq n$. Now we apply the definition of d to the following orbit segments: Set $t = K\sum_{s=1}^{n} c_s r_s$. Set $(y_0, l_0) = (w, 1)$ and then set

$$(y_i, l_i) = (x_{1j}, m_{1j}) \qquad \text{for } Kc_1(j-1) < i \leqq Kc_1 j \quad \text{and} \quad 1 \leqq j \leqq r_1,$$

$$(y_i, l_i) = (x_{2j}, m_{2j}) \text{ for } Kc_1 r_1 + Kc_2(j-1) < i \leqq Kc_1 r_1 + Kc_2 j \quad \text{and} \quad 1 \leqq j \leqq r_2$$

and so on until

$$(y_i, l_i) = (x_{nj}, m_{nj}) \qquad \text{for } K\sum_{s=1}^{n-1} c_s r_s + Kc_n(j-1) < i \leqq K\sum_{s=1}^{n-1} c_s r_s + Kc_n j \quad \text{and} \quad 1 \leqq j \leqq r_n.$$

By definition of (y_i, l_i) for $1 \leqq i \leqq t$ and of c_i and c we get

$$\sum_{i=1}^{n} \frac{q_i}{k_i} \sum_{j=1}^{r_i} S_{m_{ij}} f(x_{ij}) = \sum_{i=1}^{n} \frac{c_i}{c} \sum_{j=1}^{r_i} S_{m_{ij}} f(x_{ij}) = \frac{1}{cK} \sum_{i=1}^{t} S_{l_i} f(y_i). \tag{3.4}$$

Using also (3.2) and the definition of k_i we get

$$Kc = Kc \sum_{i=1}^{n} q_i = K \sum_{i=1}^{n} c_i k_i = K \sum_{i=1}^{n} \sum_{j=1}^{r_i} c_i m_{ij} = \sum_{i=1}^{t} l_i. \tag{3.5}$$

Furthermore the definition of D implies that

$$d(y_i, l_i, y_{i+1}, l_{i+1}, \delta) \leqq D \qquad \text{for } 0 \leqq i \leqq t \tag{3.6}$$

where $(y_{t+1}, l_{t+1}) = (y_0, l_0)$.

By the definition of d there are a periodic point $z \in X$ and integers $u_0 = 0$, $u_1, \ldots, u_t, u_{t+1} = p$ with $T^p(z) = z$ and $u_i + l_i \leqq u_{i+1}$ for $0 \leqq i \leqq t$ such that (1.1) and (1.2) hold. By (1.5) and the definition of (y_i, l_i) for $1 \leqq i \leqq t$ we have $d(y_i, l_i, y_{i+1}, l_{i+1}, \delta) \leqq \varepsilon l_i$, if $i \neq 0$ and if $i \neq K\sum_{s=1}^{h-1} c_s r_s + jc_h K$ where $1 \leqq h \leqq n$ and $1 \leqq j \leqq r_h$, that is if $(y_i, l_i) = (y_{i+1}, l_{i+1})$. Counting the number of these i's and using (3.6), (3.2), $l_0 = 1 \leqq D$ and the definitions of l_i, k_i, K, c_i and c we get from (1.1) that

$$\begin{aligned} l_0 + \sum_{i=0}^{t} (u_{i+1} - u_i - l_i) &\leqq D + \left(1 + \sum_{i=1}^{n} r_i\right) D + \sum_{i=1}^{n} \sum_{j=1}^{r_i} (c_i K - 1)\varepsilon m_{ij} \\ &\leqq \left(2 + \sum_{i=1}^{n} \sum_{j=1}^{r_i} m_{ij}\right) D + \sum_{i=1}^{n} (c_i K - 1)\varepsilon k_i \\ &\leqq \left(2 + \sum_{i=1}^{n} k_i\right) D + \sum_{i=1}^{n} c_i K \varepsilon k_i \end{aligned}$$

$$\leqq \left(2+\sum_{i=1}^{n} c_i k_i\right) D + \varepsilon K \sum_{i=1}^{n} c_i k_i$$

$$\leqq \left(2+2\sum_{i=1}^{n} c_i k_i\right) \varepsilon K$$

$$\leqq \left(2c+2c\sum_{i=1}^{n} q_i\right) \varepsilon K$$

$$= 4c\varepsilon K.$$

Hence we have shown (remark that $p = u_{t+1}$)

$$p - \sum_{i=1}^{t} l_i \leqq 4\varepsilon c K. \tag{3.7}$$

Now we get the following estimations:

$$\left| \frac{1}{cK} \sum_{i=1}^{t} S_{l_i} f(y_i) - \frac{1}{p} S_p f(z) \right| \leqq A + B + C$$

where

$$A := \left| \frac{1}{cK} \sum_{i=1}^{t} S_{l_i} f(y_i) - \frac{1}{cK} \sum_{i=1}^{t} \sum_{j=u_i}^{u_i+l_i-1} f(T^j z) \right|$$

$$\leqq \frac{1}{cK} \sum_{i=1}^{t} \sum_{j=0}^{l_i-1} |f(T^j y_i) - f(T^{u_i+j} z)|$$

$$< \frac{1}{cK} \sum_{i=1}^{t} l_i \varepsilon \quad \text{by (1.2) and (3.1)}$$

$$= \varepsilon \qquad \text{by (3.5)},$$

$$B := \left| \frac{1}{cK} \sum_{i=1}^{t} \sum_{j=u_i}^{u_i+l_i-1} f(T^j z) - \frac{1}{cK} S_p f(z) \right|$$

$$= \frac{1}{cK} \left| \sum_{j=0}^{p-1} f(T^j z) - \sum_{i=1}^{t} \sum_{j=u_i}^{u_i+l_i-1} f(T^j z) \right|$$

$$\leqq \frac{1}{cK} F \left(p - \sum_{i=1}^{t} l_i\right)$$

$$\leqq 4\varepsilon F \qquad \text{by (3.7)},$$

$$
\begin{aligned}
C &:= \left| \frac{1}{cK} S_p f(z) - \frac{1}{p} S_p f(z) \right| \\
&\leqq \left| \frac{1}{cK} - \frac{1}{p} \right| pF \\
&= \frac{F}{cK} \left| p - cK \right| \\
&= \frac{F}{cK} \left(p - \sum_{i=1}^{t} l_i \right) \quad \text{by (3.5)} \\
&\leqq 4\varepsilon F \qquad \text{by (3.7).}
\end{aligned}
$$

This together gives for all $f \in \mathscr{F}$

$$
\left| \frac{1}{cK} \sum_{i=1}^{t} S_{l_i} f(y_i) - \frac{1}{p} S_p f(z) \right| \leqq (8F + 1)\varepsilon. \tag{3.8}
$$

Now we put (3.2), (3.3) and (3.8) together, where we use (3.4) and get for all $f \in \mathscr{F}$

$$
\left| \int f d\mu - \frac{1}{p} S_p f(z) \right| < (10F + 3)\varepsilon = \eta.
$$

This implies that $m_z \in W(\mu, \mathscr{F}, \eta) \subset U$. Furthermore $\operatorname{dist}(z, w) < \delta$ by (1.2), since we have chosen $(y_0, l_0) = (w, 1)$. Hence $z \in V$ by the choice of δ. This proves Theorem 1.

§4. Unimodal and monotonic mod 1 transformations

We summarize first some results which are shown in [2] and [3]. Let (X, T) be a piecewise monotonic dynamical system as introduced in [3], that is, the totally ordered set X has a finite partition $\mathscr{Z}$ into closed-open intervals such that $T \mid Z$ is monotone and $T(Z)$ is again an interval for all $Z \in \mathscr{Z}$. If one has a piecewise monotonic map T on $[0, 1]$, i.e. there are points $c_0 = 0 < c_1 < \cdots < c_n = 1$ such that $T \mid (c_i, c_{i+1})$ is continuous and monotone for $0 \leqq i < n$, one has to substitute all $y \in (0, 1)$ with $T^k(y) = c_i$ for some i and some $k \geqq 0$ by two points $y-$ and $y+$ and to extend T continuously with respect to the order topology, in order to get a piecewise monotonic dynamical system as defined above. These countably many points, which are added, can contain only finitely many periodic points, so that the validity of the assertion of Theorem 1 for an invariant subset of ([0, 1], T) with no isolated points is not affected by this construction. We number the elements of $\mathscr{Z}$ according to their order in X such that $\mathscr{Z} = \{Z_1, Z_2, \ldots, Z_n\}$ and $Z_i = [c_{i-1}+, c_i-]$. The symbolic expansion φ: $X \to S^{\mathbf{N}}$, where $S = \{1, 2, \ldots, n\}$ is defined as

$$\varphi(x) = x_1x_2x_3\cdots, \qquad \text{where } x_i \text{ is such that } T^{i-1}(x) \in Z_{x_i}.$$

Since the elements of $\mathscr{Z}$ are closed, $Y := \varphi(X) \subset S^{\mathbf{N}}$ is closed. Furthermore $\varphi \circ T = \sigma \circ \varphi$. Hence Y is σ-invariant and (Y, σ) is a shift space. If $\mathscr{Z}$ is a generator then (X, T) and (Y, σ) are topologically isomorphic. Hence it suffices to show the specification property for topologically transitive subsets $\varphi(L)$ of (Y, σ), where L is one of the topologically transitive subsets of (X, T) described in the introduction. If $\mathscr{Z}$ is not a generator, then φ collapses certain intervals to single points. But one easily shows that the assertion of Theorem 1 holds for a topologically transitive subset L of (X, T), if it holds for the corresponding topologically transitive subset $L' := \varphi(L)$ of (Y, σ), since for every $\delta > 0$ there are only finitely many periodic points $z \in Y$ such that the interval $\varphi^{-1}(\{z\})$ has length $> \delta$. So it suffices also in the case, where $\mathscr{Z}$ is not a generator, to consider (Y, σ) instead of (X, T).

For a piecewise monotonic dynamical system (X, T) there exists an oriented graph $\mathscr{D}$ and a map Ψ: $\mathscr{D} \to S$ such that Y is the set of all $\Psi(C_1)\Psi(C_2)\cdots$, where $C_1C_2\cdots$ is a path in $\mathscr{D}$. (In some of the papers we cite, the elements of S are not assigned to the elements of $\mathscr{D}$, but to the arrows in $\mathscr{D}$. This makes no difference.) For a topologically transitive subset L' of Y with positive entropy, which corresponds to one of the topologically transitive subsets L of (X, T) considered in the introduction, there is an irreducible subset $\mathscr{C}$ of $\mathscr{D}$, such that L' is the set of all $\Psi(C_1)\Psi(C_2)\cdots$, where $C_1C_2\cdots$ is a path in $\mathscr{C}$. Irreducibility of $\mathscr{C}$ means here, that for all $C, D \in \mathscr{C}$ there is a path from $\mathscr{C}$ to D and that every subset of $\mathscr{D}$ containing $\mathscr{C}$ strictly does not have this property (cf. theorem 11 of [3]). Hence in order to prove Theorem 2, it suffices to show that every path in $\mathscr{C}$ satisfies either (i) or (ii) of Lemma 1, where $\mathscr{C}$ is an irreducible subset of the oriented graph $\mathscr{D}$ of one of the transformations considered in Theorem 2. We do this in the rest of this section.

By Theorem 1 of [6], the oriented graph $\mathscr{D}$ of a unimodal map is either finite, a case considered at the end of §2, or $\mathscr{D} = \{A_i: i \geqq 1\}$ and there is a sequence $(r_i)_{i \geqq 1}$ of integers $\geqq 1$ with $r_1 = 1$ such that we have the following arrows in $\mathscr{D}$, where $R_j = r_1 + r_2 + \cdots + r_j$:

$$(4.1) \qquad A_i \to A_{i-1} \quad \text{for } i \geqq 1, \qquad A_{R_j} \to A_{r_j} \quad \text{for } j \geqq 1.$$

Let $e_1e_2\cdots\in\{1,2\}^{\mathbf{N}}$ be the symbolic expansion $\varphi(1)$ of 1. Set $\Psi(A_i)=e_i$ for $i\geqq 1$. This gives then the shift space (Y,σ) (cf. [6]). We need two more facts about the sequence $(r_i)_{i\geqq 1}$ (cf. (2.3) of [6] and lemma 1 of [5]). If $k\geqq 1$ then

$$e_{R_k+1+j}=e_j \qquad \text{for } 1\leqq j\leqq r_{k+1}-1 \tag{4.2}$$

and if we set $R_0=0$, we have for $i\geqq 1$ that

$$\text{there is a } P(i) \qquad \text{with } r_i=R_{P(i)}+1 \quad \text{and} \quad 0\leqq P(i)\leqq i-1. \tag{4.3}$$

Now we consider an irreducible subset $\mathscr{C}$ of $\mathscr{D}$. By (4.1) there are $u<v\leqq\infty$ such that $\mathscr{C}=\{A_i: u\leqq i<v\}$. If $v<\infty$ then $\mathscr{C}$ is finite, a case we have considered at the end of §2. Hence we assume $v=\infty$. Then by (4.1) there is a path from every $D\in\mathscr{D}$ to $\mathscr{C}$. By definition of irreducibility, we get for $D\in\mathscr{D}$ that

$$\text{if there is a path from } \mathscr{C} \text{ to } D \text{ then } D\in\mathscr{C}. \tag{4.4}$$

We apply Lemma 1. To this end let $C_1C_2\cdots$ be a path in $\mathscr{C}$. Choose $i_0<i_1<i_2<\cdots$ such that $C_{i_0}, C_{i_1},\ldots$ are all elements of $\{A_{R_j}: j\geqq 1\}$, which occur in $C_1C_2\cdots$. By (4.1) and (4.3), for every k there is a j such that $C_{i_k+1}C_{i_k+2}\cdots C_{i_{k+1}}$ is $A_{R_j+1}A_{R_j+2}\cdots A_{R_{j+1}}$. In particular, $i_{k+1}-i_k=r_{j+1}$. If $i_{k+1}-i_k$ does not tend to ∞, then there is a $t\in\mathbf{N}$ with $i_{k+1}-i_k=t$ for infinitely many k. But then A_t is a successor of $C_{i_{k+1}}=A_{R_{j+1}}$ for these infinitely many k by (4.1) since $r_{j+1}=t$. By (4.4) we get $A_t\in\mathscr{C}$, and (i) of Lemma 1 holds with $D=A_t$.

Therefore, if (i) of Lemma 1 does not hold, we have $i_{k+1}-i_k\to\infty$ for $k\to\infty$. We shall show that for every j with $A_{R_{j+1}}\in\mathscr{C}$ there is a closed path $D_1D_2\cdots D_{r_{j+1}}D_1D_2\cdots$ in $\mathscr{C}$ with

$$\Psi(D_m)=\Psi(A_{R_j+1+m}) \qquad \text{for } 1\leqq m\leqq r_{j+1}-1. \tag{4.5}$$

As for every k there is a j with $C_{i_{k-1}+2}C_{i_{k-1}+3}\cdots C_{i_k}=A_{R_j+2}A_{R_j+3}\cdots A_{R_{j+1}}$ and $A_{R_{j+1}}=C_{i_k}\in\mathscr{C}$, this gives (ii) of Lemma 1 for $c=1$, $l=r_{j+1}$, $N(k)=k-1$ for all k, and for the sequence $(i_k+1)_{k\geqq 0}$ instead of $(i_k)_{k\geqq 0}$.

It remains to show (4.5). Set $u=P(j+1)$, that is, $r_{j+1}=R_u+1$ by (4.3), and set $D_1D_2\cdots D_{r_{j+1}}=A_{R_u+2}\cdots A_{R_{u+1}}A_{r_{u+1}}\cdots A_{R_u+1}$. We have $r_{u+1}\leqq R_u+1$ by (4.3). Hence $D_1D_2\cdots D_{r_{j+1}}D_1D_2\cdots$ is a closed path by (4.1), which is in $\mathscr{C}$ by (4.4), since $A_{r_{j+1}}=A_{R_u+1}$ is a successor of $A_{R_{j+1}}\in\mathscr{C}$ (cf. (4.1)) and hence there is a path from $\mathscr{C}$ to every D_i. Furthermore, by definition of Ψ and by (4.2) we get $\Psi(A_{R_u+1+m})=e_m$ for $1\leqq m\leqq r_{u+1}-1$ and $\Psi(A_m)=e_m$ for $r_{u+1}\leqq m\leqq R_u+1=r_{j+1}$. Hence $\Psi(D_m)=e_m$ for $1\leqq m\leqq r_{j+1}$. Similarly one gets

$\Psi(A_{R_j+1+m}) = e_m$ for $1 \leqq m \leqq r_{j+1} - 1$ and hence (4.5) follows. This finishes the proof of Theorem 2 for unimodel maps.

Now we consider the map $T(x) = f(x) \bmod 1$ on $[0, 1)$, where f: $[0, 1) \rightarrow \mathbf{R}$ is inceasing, $f(0) \in [0, 1)$ and $f(1-) \in (1, 2]$. In this case the shiftspace (Y, σ) introduced above is determined by $\boldsymbol{a} = a_1a_2a_3 \cdots = \varphi(0)$ and $\boldsymbol{b} = b_1b_2b_3 \cdots = \varphi(1-)$ (cf. (1.2) of [9]). We can exclude that $\sigma^k\boldsymbol{a} = \boldsymbol{b}$ or that $\sigma^k\boldsymbol{b} = \boldsymbol{a}$ for some k. If $\sigma^k\boldsymbol{a} = \boldsymbol{b}$, we cancel from (Y, σ) the set $\bigcup_{i=0}^{\infty} \sigma^{-k}\{\boldsymbol{a}\}$, which is wandering except the orbit of $\boldsymbol{a}$, if $\boldsymbol{a}$ is periodic. In this case the orbit of $\boldsymbol{a}$ is isolated (cf. Lemma 8 of [9]). Hence we have not changed the topologically transitive subsets L, which we consider. By this we get a shiftspace (Y', σ) of the same form as (Y, σ) determined by

$$\boldsymbol{a}' = a_1a_2 \cdots a_{k-1}2a_1a_2 \cdots a_{k-1}2a_1a_2 \cdots \quad \text{and} \quad \boldsymbol{b}' = \boldsymbol{b}.$$

Similar arguments apply if $\sigma^k\boldsymbol{b} = \boldsymbol{a}$. assuming $\sigma^k\boldsymbol{a} \neq \boldsymbol{b}$ and $\sigma^k\boldsymbol{b} \neq \boldsymbol{a}$ for all k, by theorem 1 of [9] the oriented graph $\mathscr{D}$ is given by $\{A_i, B_i: i \geqq 1\}$, and there are sequences $(r_i)_{i \geqq 1}$ and $(s_i)_{i \geqq 1}$ of integers $\geqq 1$, such that we have the following arrows in $\mathscr{D}$, where we set $R_j = r_1 + r_2 + \cdots + r_j$ and $S_j = s_1 + s_2 + \cdots + s_j$:

$$\begin{aligned} &A_i \rightarrow A_{i+1} \quad B_i \rightarrow B_{i+1} \quad \text{for } i \geqq 1, \\ &A_{R_j} \rightarrow B_{r_j}, \quad B_{S_j} \rightarrow A_{s_j} \quad \text{for } j \geqq 1, \end{aligned} \tag{4.6}$$

The shift space (Y, σ) is then given by Ψ: $\mathscr{D} \rightarrow \{1, 2\}^{\mathbf{N}}$, where

$$\Psi(A_i) = a_i, \quad \Psi(B_i) = b_i \qquad \text{for } i \geqq 1. \tag{4.7}$$

By (1.3) and (1.4) of [9] we know for $j \geqq 1$

$$\begin{aligned} a_{R_j+1+i} &= b_i \qquad \text{for } 1 \leqq j \leqq r_{j+1} - 1, \\ b_{S_j+1+i} &= a_i \qquad \text{for } 1 \leqq j \leqq s_{j+1} - 1. \end{aligned} \tag{4.8}$$

For $i \geqq 1$, lemma 1 of [7] says, where we set $R_0 = S_0 = 0$,

$$\begin{aligned} &\text{there is a } P(i) \geqq 0 \qquad \text{with } r_i = S_{P(i)} + 1, \\ &\text{there is a } Q(i) \geqq 0 \qquad \text{with } s_i = R_{Q(i)} + 1. \end{aligned} \tag{4.9}$$

The irreducible subsets $\mathscr{C}$ of $\mathscr{D}$ are investigated in [9]. Six cases are considered there. Cases (b) and (d) occur only if $\sigma^k\boldsymbol{a} = \boldsymbol{b}$ or $\sigma^k\boldsymbol{b} = \boldsymbol{a}$ for some k, a case which we can exclude. In cases (c), (e) and (f), $\mathscr{D}$ has only finite irreducible subsets, which we have already considered in §2. So case (a) remains. In this case there is only one infinite $\mathscr{C}$, for which there are $u, v \in \mathbf{N}$

such that $\mathscr{C} = \{A_i, B_j: i \geqq u, j \geqq v\}$. For this $\mathscr{C}$ we get for $D \in \mathscr{D}$ in the same way as for unimodel maps that

(4.10) if there is a path from $\mathscr{C}$ to D then $D \in \mathscr{C}$.

Now we can show the assumptions of Lemma 1. To this end let $C_1C_2\cdots$ be a path in $\mathscr{C}$. Choose $i_0 < i_1 < \cdots$ such that $C_{i_0}, C_{i_1}, \ldots$ are all elements of $\{A_{R_j}, B_{S_j}: j \geqq 1\}$, which occur in $C_1C_2\cdots$. By (4.6) and (4.9), for every k there is a j such that $C_{i_k+1}C_{i_k+2}\cdots C_{i_{k+1}}$ is either $A_{R_j+1}A_{R_j+2}\cdots A_{R_{j+1}}$ or $B_{S_j+1}B_{S_j+2}\cdots B_{S_{j+1}}$. In particular $i_{k+1} - i_k = r_{j+1}$ or $i_{k+1} - i_k = s_{j+1}$. If $i_{k+1} - i_k$ does not tend to infinity, then there is a $t \in \mathbf{N}$ with $i_{k+1} - i_k = t$ for infinitely many k. Then there are infinitely many k such that $i_{k+1} - i_k = r_{j+1} = t$ (or $i_{k+1} - i_k = s_{j+1} = t$) and hence by (4.6) B_t (or A_t) is a successor of $C_{i_{k+1}} = A_{R_{j+1}}$ (or $C_{i_{k+1}} = B_{S_{j+1}}$) for these infinitely many k. By (4.10), $B_t \in \mathscr{C}$ (or $A_t \in \mathscr{C}$) and (i) of Lemma 1 holds with $D = B_t$ (or $D = A_t$).

Therefore, if (i) of Lemma 1 does not hold, we have $i_k - i_{k-1} \to \infty$. Suppose $C_{i_{k-1}+1}C_{i_{k-1}+2}\cdots C_{i_k} = A_{R_j+1}A_{R_j+2}\cdots A_{R_{j+1}}$. If it is $B_{S_j+1}B_{S_j+2}\cdots B_{S_{j+1}}$ the proof is analogous. We shall find a closed path $D_1D_2\cdots D_lD_1D_2\cdots$ in $\mathscr{C}$ such that either

$$(4.11) \qquad l = r_{j+1} \quad \text{and} \quad \Psi(D_m) = \Psi(A_{R_j+1+m}) \qquad \text{for } 1 \leqq m < r_{j+1}$$

or

$$(4.12) \qquad l = R_{j+1} + 1 \quad \text{and} \quad \Psi(D_m) = \Psi(A_m) \qquad \text{for } 1 \leqq m \leqq R_{j+1}.$$

We show that then (ii) of Lemma 1 holds for the sequence $(i_k + 1)_{k \geqq 1}$ instead of $(i_k)_{k \geqq 1}$ and with $c = 1 + R_u$, if $C_{i_0+1} = A_{R_u+1}$, and with $c = 1 + S_u$, if $C_{i_0+1} = B_{S_u+1}$. If (4.11) holds, we set $N(k) = k - 1$ and get (ii) of Lemma 1 from (4.11), as $c \geqq 1$ and $l = r_{j+1} = i_k - i_{k-1}$. Now assume that (4.12) holds. We consider first the case where an $N(k) \leqq k - 1$ exists such that

$$C_{i_{k-m-1}+1}\cdots C_{i_{k-m}} = A_{R_{j-m}+1}\cdots A_{R_{j-m+1}} \qquad \text{for } 0 \leqq m \leqq k - N(k) - 2,$$

$$C_{i_{N(k)}+1}\cdots C_{i_{N(k)+1}} = B_{S_n+1}\cdots B_{S_{n+1}} \qquad \text{for some } n.$$

As $A_{R_{j-k+N(k)+2}+1} = C_{i_{N(k)+1}+1}$ is a successor of $B_{S_{n+1}} = C_{i_{N(k)+1}}$, we get $s_{n+1} = R_{j-k+N(k)+2} + 1$ by (4.6). As $\Psi(B_{S_n+1+m}) = b_{S_n+1+m} = a_m = \Psi(A_m)$ for $1 \leqq m \leqq s_{n+1} - 1$ by (4.7) and (4.8), we get by (4.12) that $\Psi(D_m) = \Psi(C_{i_{N(k)}+1+m})$ for $1 \leqq m \leqq R_{j+1} = i_k - i_{N(k)} - 1$. This implies (ii) of Lemma 1, as $c \geqq 1$. That $N(k)$ does not exist, means that $j \geqq k - 1$ and that

$$C_{i_{k-m-1}+1}\cdots C_{i_{k-m}} = A_{R_{j-m}+1}\cdots A_{R_{j-m+1}} \qquad \text{for } 0 \leqq m \leqq k - 1.$$

In this case we set $N(k) = 0$ and use the closed path

$$D_{R_{j-k+1}+1} \cdots D_{R_{j+1}+1} D_1 \cdots D_{R_{j-k+1}} D_{R_{j-k+1}+1} \cdots.$$

Then (4.12) implies (ii) of Lemma 1, as $C_{i_0+1} = A_{R_{j-k+1}+1}$ gives $c = R_{j-k+1} + 1$, as $l = R_{j+1} + 1$, and as $i_k - i_{N(k)} = i_k - i_0 = R_{j+1} - R_{j-k+1}$.

It remains to show the existence of a closed path $D_1 D_2 \cdots D_l D_1 D_2 \cdots$ in $\mathscr{C}$ such that either (4.11) or (4.12) holds. We suppose first that there is a $t \geqq j + 1$ with

$$(4.13) \qquad r_{j+m} = r_{j+1} \quad \text{for } 1 \leqq m \leqq t - j \quad \text{and} \quad r_{t+1} < r_{j+1}$$

and show that (4.11) holds. Set $n = Q(P(t))$, that is, $s_{P(t)} = 1 + R_n$ by (4.9). By lemma 2 of [8] and by (4.13) we have $r_{n+1} \leqq r_{t+1} < r_{j+1} = r_t$. Set

$$l = r_{j+1} = r_t \quad \text{and} \quad D_1 \cdots D_l = A_{R_n+1} \cdots A_{R_{n+1}} B_{r_{n+1}} \cdots B_{r_t - 1}.$$

Then $D_1 D_2 \cdots D_l D_1 D_2 \cdots$ is a closed path by (4.6), as $r_t - 1 = S_{P(t)}$ (cf. (4.9)), as $R_n + 1 = s_{P(t)}$, and as $r_{n+1} \leqq r_t - 1$. By (4.6) there is a path from $A_{R_{j+1}} = C_{i_k} \in \mathscr{C}$ to $A_{R_{t+1}}$ as $t + 1 > j + 1$, and $B_{r_{t+1}}$ is a successor of $A_{R_{t+1}}$. Since $r_{n+1} \leqq r_{t+1} \leqq r_t - 1$, there is an m with $1 \leqq m \leqq l$ and $B_{r_{t+1}} = D_m$. Hence there is a path from $\mathscr{C}$ to one and hence to all D_i. By (4.10) we get $D_i \in \mathscr{C}$ for all i. Now we show (4.11), that is, $\Psi(D_m) = \Psi(A_{R_j+1+m})$ for $1 \leqq m < r_{j+1}$. This follows from $\Psi(A_{R_n+1+m}) = a_{R_n+m+1} = b_m$ for $1 \leqq m \leqq r_{n+1} - 1$, which holds by (4.7) and (4.8), from $\Psi(B_m) = b_m$ for $r_{n+1} \leqq m < r_t = r_{j+1}$, which holds by (4.7), and from $\Psi(A_{R_j+1+m}) = a_{R_j+1+m} = b_m$ for $1 \leqq m < r_{j+1}$, which holds by (4.7) and (4.8).

Now we suppose that (4.13) does not hold. Set $n = P(j + 1)$ and suppose $s_{n+1} \geqq 1 + R_{j+1}$. If $s_{n+1} > 1 + R_{j+1}$, the requirements of Lemma 3 of [8] are satisfied using also the contrary of (4.13). If $s_{n+1} = 1 + R_{j+1}$, that is, $Q(n + 1) = j + 1$, then $P(Q(n + 1)) = n$ by definition of n. By lemma 2 of [8], there is a $u \leqq \infty$ with $s_{n+1+i} = s_{n+i}$ for $1 \leqq i < u$ and $s_{n+1+u} > s_{n+u}$, if $u < \infty$. Together with the contrary of (4.13) we get again the requirements of lemma 3 of [8]. Hence, if $s_{n+1} \geqq 1 + R_{j+1}$, this lemma implies that $r_{j+i} \geqq r_{j+1}$ and $s_{n+i} \geqq s_{n+1}$ for all $i \geqq 1$. Then, by (4.6), there is no arrow from $\mathscr{C}' := \{A_i, B_m: i \geqq R_{j+1} + 1, m \geqq S_n + 1\}$ to $\mathscr{C} \setminus \mathscr{C}'$. Furthermore $A_{R_{j+1}} = C_{i_k} \in \mathscr{C} \setminus \mathscr{C}'$, hence $\mathscr{C} \setminus \mathscr{C}' \neq \varnothing$. This contradicts the irreducibility of $\mathscr{C}$. Hence $s_{n+1} < 1 + R_{j+1}$, which implies $s_{n+1} \leqq 1 + R_j$ by (4.9). Now we can show (4.12). Set

$$l = 1 + R_{j+1} \quad \text{and} \quad D_1 D_2 \cdots D_l = B_{S_n+2} \cdots B_{S_{n+1}} A_{s_{n+1}} \cdots A_{R_{j+1}} B_{r_{j+1}}.$$

Then $D_1D_2 \cdots D_lD_1D_2 \cdots$ is a closed path, as $s_{n+1} \leqq 1 + R_j \leqq R_{j+1}$ and $n = P(j+1)$, that is, $r_{j+1} = 1 + S_n$ by (4.9). By (4.10), this path is in $\mathscr{C}$, as it contains $A_{R_{j+1}} = C_{i_k} \in \mathscr{C}$. It remains to show that $\Psi(D_m) = \Psi(A_m)$ for $1 \leqq m \leqq R_{j+1}$. This follows from $\Psi(B_{S_n+m+1}) = b_{S_n+m+1} = a_m = \Psi(A_m)$ for $1 \leqq m \leqq s_{n+1} - 1$, which holds by (4.7) and (4.8). This completes the proof of (4.12) and also the proof of Theorem 2.

References

1. M. Denker, C. Grillenberger and K. Sigmund, *Ergodic theory on compact spaces*, Lecture Notes in Math. **527**, Springer-Verlag, Berlin, 1976.

2. F. Hofbauer, *The structure of piecewise monotonic transformations*, Ergodic Theory & Dynamical Systems **1** (1981), 159–178.

3. F. Hofbauer, *Piecewise invertible dynamical systems*, Probab. Th. Rel. Fields **72** (1986), 359–386.

4. F. Hofbauer, *β-shifts have unique maximal measure*, Monatsh. Math. **85** (1978), 189–198.

5. F. Hofbauer, *The topological entropy of the transformation $x \to ax(1-x)$*, Monatsh. Math. **90** (1980), 117–141.

6. F. Hofbauer, *Kneading invariants and Markov diagrams*, in *Ergodic Theory and Related Topics. Proceedings* (H. Michel, ed.), Akademie-Verlag, Berlin, 1982.

7. F. Hofbauer, *Maximal measures for simple piecewise monotonic transformations*, Z. Wahrscheinlichkeitstheor. Verw. Geb. **52** (1980), 289–300.

8. F. Hofbauer, *The maximal measure for linear mod one transformations*, J. London Math. Soc. **23** (1981), 92–112.

9. F. Hofbauer, *Monotonic mod one transformations*, Studia Math. **80** (1984), 17–40.

10. K. Sigmund, *Generic properties of invariant measures for axiom A-diffeomorphisms*, Invent. Math. **11** (1970), 99–109.

11. K. Sigmund, *On the distribution of periodic points for β-shifts*, Monatsh. Math. **82** (1976), 247–252.